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Short Communication

Vibrations of a plate with an attached two degree of freedom system

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Abstract

This note deals with a theoretical analysis of the dynamical behavior of a system made up of a plate with a two degree of freedom (2-dof) system elastically mounted. This study was performed by means of an analytical model based on Lagrange's multipliers. The results are verified with the values obtained using FEM.

The analysis is of interest from both academic and technological viewpoints. The case of a plate structure supporting a 2-dof system has not been previously considered in the technical literature using the Lagrange's multipliers approach. This system can be considered as a dynamic absorber.

1. Introduction

The study of the dynamics of plates or slabs is of fundamental importance in structural design. In many everyday situations the plate supports a discrete mass–spring system. The problem of free vibrations of beams with additional complicating elements (elastic supports, elastically attached

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masses, etc.) has been extensively studied and hence only a few selected references are given here [1–5].

The analysis of higher-order discrete systems attached to continuous structural elements has not, apparently, been performed in the case of plates. It is the goal of the present investigation to consider an attached two degree of freedom (2-dof) which increases substantially the scope of previously developed analysis.

2-dof systems added to structural elements are more difficult to treat than single dof systems, specially because of the complexity of the mathematical expressions. An interesting approach to the problem, in the case of beam vibrations was given by Jen and Magrab [6], and more recently in an important paper by Wu and Whittaker [7].

Here, a well known but not usually employed method, previously developed by Dowell [2] to solve vibrations of beams carrying elastically mounted systems, is employed for the title problem. The eigenfrequencies and normal mode shapes of the system will be obtained for two types of boundary conditions.

In Section 2, the analytical formulation of the problem is presented and in Section 3, eigenfrequencies are computed using the finite element method (FEM). The results are shown in Section 4 where we made a comparison between these two approaches (analytical and FEM). Concluding remarks are presented in Section 5.

2. Lagrange's formulation

Fig. 1 shows the 2-dof spring–mass system attached to a plate. The intervening parameters m_e, I_e, k_i are, respectively, the lumped mass, mass moment of inertia, and spring constants of the 2-dof spring–mass system, a_1 and a_2 are, respectively, the distances between the mass center and the two springs k_1 and k_2.

Initially, the plate and the 2-dof spring–mass system are considered to be unconnected. The total kinetic and strain energies of the entire system are

$$T = \frac{1}{2}\sum_{i,j}^{n,n'} m_{ij}\dot{q}_{ij}^2 + \frac{1}{2}\frac{m_e}{(a_1+a_2)^2}(\dot{z}_{m1}a_2 + \dot{z}_{m2}a_1)^2 + \frac{1}{2}\frac{I_e}{(a_1+a_2)^2}(\dot{z}_{m2} - \dot{z}_{m1})^2, \tag{1}$$

$$V = \frac{1}{2}\sum_{i,j}^{n,n'} m_{ij}\omega_{ij}^2 q_{ij}^2 + \frac{1}{2}k_1(z_{m1} - z_1)^2 + \frac{1}{2}k_2(z_{m2} - z_2)^2, \tag{2}$$

where $q_{i,j}$ are the generalized coordinates, ω_{ij} are the eigenfrequencies of the bare plate and the m_{ij}'s are given by

$$\rho h \int_\Omega \phi_{ij}\phi_{mn}\,\mathrm{d}\Omega = \begin{cases} 0 & \text{if } i \neq m \text{ or } j, \\ m_{ij} & \text{if } i = m \text{ and } j = n. \end{cases} \tag{3}$$

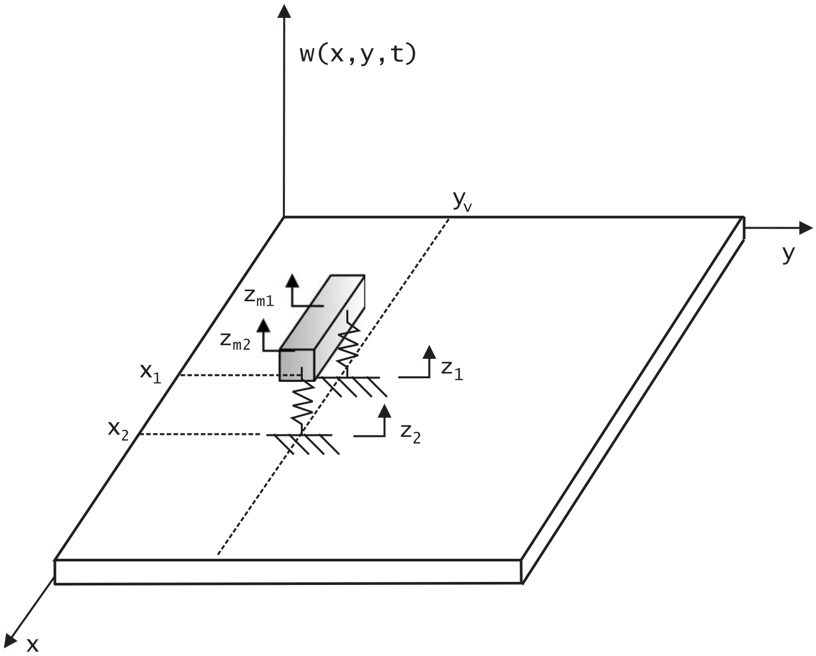

Fig. 1. Plate with the 2-dof system attached to it.

The transverse displacement of the plate is represented by

$$w(x,y,t) = \sum_{i,j}^{n,n'} q_{ij}(t)\phi_{ij}(x,y), \tag{4}$$

where the $\phi_{ij}(x,y)$ are the normal mode shapes of the "bare" plate. For convenience, the term "bare" plate will be used to define the primitive continuous system or in other words: the original structural element without altering the discrete system.

As it is shown in Fig. 1, the plate is constrained at the points x_1, y_v and x_2, y_v by the 2-dof spring–mass system. This can be expressed in the form

$$f_1 = \sum_{i,j}^{n,n'} q_{ij}(t)\phi_{ij}(x_1,y_v) - z_1 = 0,$$

$$f_2 = \sum_{i,j}^{n,n'} q_{ij}(t)\phi_{ij}(x_2,y_v) - z_2 = 0. \tag{5}$$

The equations of motion will be obtained by means of Lagrange's equations. Additionally, we must add the restrictions into them, given by Eqs. (5), by means of Lagrange multipliers. So, for a system with N dof (here $N = n \times n'$) in which v are redundant coordinates, the equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial s_\kappa}\right) - \frac{\partial L}{\partial s_\kappa} = \sum_{l=1}^{v} \lambda_l \frac{\partial f_l}{\partial s_\kappa}, \quad \kappa = 1,\ldots,N+v+2 \tag{6}$$

where the λ_l are the Lagrange multipliers. With the Lagrangian

$$L = T - V, \tag{7}$$

upon taking into account the correspondence

$$[s_1, \ldots, s_N, s_{N+1}, \ldots, s_{N+4}] \equiv [q_{11}, \ldots, q_{nn'}, z_{m1}, z_{m2}, z_1, z_2], \tag{8}$$

using Lagrange's equations (6) yields

$$\begin{aligned}
&m_{ij}(\ddot{q}_{ij} + \omega_{ij}^2 q_{ij}) = \lambda_1 \phi_{ij}(x_1, y_v) + \lambda_2 \phi_{ij}(x_2, y_v),\\
&\frac{m_e}{(a_1 + a_2)^2} a_2(\ddot{z}_{m1} a_2 + \ddot{z}_{m2} a_1) - \frac{I_e}{(a_1 + a_2)^2}(\ddot{z}_{m2} - \ddot{z}_{m1}) + k_1(z_{m1} - z_1) = 0,\\
&\frac{m_e}{(a_1 + a_2)^2} a_1(\ddot{z}_{m1} a_2 + \ddot{z}_{m2} a_1) + \frac{I_e}{(a_1 + a_2)^2}(\ddot{z}_{m2} - \ddot{z}_{m1}) + k_2(z_{m2} - z_2) = 0,\\
&k_1(z_{m1} - z_1) = \lambda_1, \quad k_2(z_{m2} - z_2) = \lambda_2.
\end{aligned} \tag{9}$$

Assuming simple harmonic motion the variables become

$$q_{ij}(t) = \bar{q}_{ij} \mathrm{e}^{\mathrm{i}\omega t}, \quad z_{m1}(t) = \bar{z}_{m1} \mathrm{e}^{\mathrm{i}\omega t}, \quad z_{m2}(t) = \bar{z}_{m2} \mathrm{e}^{\mathrm{i}\omega t}, \quad z_1(t) = \bar{z}_1 \mathrm{e}^{\mathrm{i}\omega t}, \quad z_2(t) = \bar{z}_2 \mathrm{e}^{\mathrm{i}\omega t}.$$

Computing q_{ij}, z_{m1}, z_{m2}, z_1 and z_2 in terms of λ_1 and λ_2 and substituting the result into system (5) one obtains the matrix equation

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{10}$$

where the α's are given by

$$\alpha_{11} = \sum_{i,j}^{n,n'} \frac{\phi_{ij}^2(x_1, y_v)}{m_{ij}(\omega_{ij}^2 - \omega^2)} + \frac{1}{k_1} - \frac{I_e + m_e a_1^2}{m_e I_e \omega^2}, \tag{11}$$

$$\alpha_{12} = \sum_{i,j}^{n,n'} \frac{\phi_{ij}(x_1, y_v)\phi_{ij}(x_2, y_v)}{m_{ij}(\omega_{ij}^2 - \omega^2)} - \frac{I_e - m_e a_1 a_2}{m_e I_e \omega^2}, \tag{12}$$

$$\alpha_{21} = \sum_{i,j}^{n,n'} \frac{\phi_{ij}(x_2, y_v)\phi_{ij}(x_1, y_v)}{m_{ij}(\omega_{ij}^2 - \omega^2)} - \frac{I_e - m_e a_1 a_2}{m_e I_e \omega^2}, \tag{13}$$

$$\alpha_{22} = \sum_{i,j}^{n,n'} \frac{\phi_{ij}^2(x_2, y_v)}{m_{ij}(\omega_{ij}^2 - \omega^2)} + \frac{1}{k_2} - \frac{I_e + m_e a_2^2}{m_e I_e \omega^2}. \tag{14}$$

From the matrix equation, the eigenfrequencies of the problem can be obtained, if the determinant of the coefficient matrix is equated to zero.

Once the eigenvalues (frequencies) are calculated, the eigenvectors ($\lambda_{1,2}$) are obtained in a straightforward fashion. From Eq. (9), the $\bar{q}'_{ij}$s can be expressed as

$$\bar{q}_{ij}^{(k)} = \frac{\lambda_1^{(k)}\phi_{ij}(x_1, y_v) + \lambda_2^{(k)}\phi_{ij}(x_2, y_v)}{m_{ij}(\omega_{ij}^2 - \omega^{(k)^2})}, \tag{15}$$

where the superscript (k) labels the eigenvalue under consideration.

Now, we are in a condition to express the new modes of the plate carrying a 2-dof mass–spring system:

$$Y^{(k)}(x, y) = \sum_{i=1,j=1}^{n,n'} \phi_{ij}(x, y)\bar{q}_{ij}^{(k)}, \quad k = 1, \ldots, N + 2. \tag{16}$$

3. Finite element formulation

According to the classical thin-plate theory, the governing equation for the transverse displacement of the plate $w(x, y, t)$ is the well-known elliptic, fourth-order, partial differential equation

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = f, \quad \text{in } \Omega \times [0, t], \tag{17}$$

where ∇^4 is the two-dimensional bi-harmonic operator, h the thickness of the plate, D the flexural rigidity (i.e. $D = Eh^3/12(1 - \nu^2)$), ρ the mass density, f the lateral load per unit area and Ω the plate domain with boundary $\partial\Omega$. The plate discretization uses rectangular elements. In order to satisfy C^1 continuity of w and conformal requirements, the bicubic Bogner–Fox–Schmidt element was used (see Ref. [9]). Nodal dofs for this element are

$$u^e = \left\{ w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y} \right\}. \tag{18}$$

Eigenfrequencies are calculated solving the so-called characteristic equation

$$\det(-\omega^2 \mathbf{M} + \mathbf{K}) = 0, \tag{19}$$

where the global mass and stiffness matrices $\mathbf{M}$ and $\mathbf{K}$ are given by

$$\mathbf{M} = \mathop{\mathsf{A}}_{e=1}^{nelem} h \int_{\Omega^e} \rho \mathbf{N}_e^{\mathrm{T}} \mathbf{N}_e \, \mathrm{d}\Omega^e, \tag{20}$$

$$\mathbf{K} = \mathop{\mathsf{A}}_{e=1}^{nelem} \int_{\Omega^e} \mathbf{B}_e^{\mathrm{T}} \mathbf{D} \mathbf{B}_e \, \mathrm{d}\Omega^e, \tag{21}$$

A being the standard finite element assembly operator and *nelem* the number of mesh element. The constitutive matrix $\mathbf{D}$ for the plate is

$$\mathbf{D} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{22}$$

and the strain–displacement matrix $\mathbf{B}_e$ is obtained by differentiation of the shape functions $\mathbf{N}_e$:

$$\mathbf{B}_e = \begin{bmatrix} -\dfrac{\partial^2 N_e}{\partial x^2} \\ -\dfrac{\partial^2 N_e}{\partial y^2} \\ -\dfrac{2\partial^2 N_e}{\partial x \partial y} \end{bmatrix}. \tag{23}$$

The 2-dof spring–mass system is treated as in Ref. [7] where the spring–mass system is attached to a prismatic beam. The mass and stiffness matrices for the 2-dof spring–mass element that have to be assembled are

$$\mathbf{m}_{\mathrm{sm}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_e & 0 \\ 0 & 0 & 0 & I_e \end{bmatrix}, \tag{24}$$

$$\mathbf{k}_{\mathrm{sm}} = k \begin{bmatrix} 1 & 0 & -1 & a_1 \\ 0 & 1 & -1 & -a_2 \\ -1 & -1 & 2 & (a_2 - a_1) \\ a_1 & -a_2 & (a_2 - a_1) & (a_1 + a_2) \end{bmatrix}, \tag{25}$$

with associated dof's $u^e_{\mathrm{sm}} = \{w_i, w_k, w_{\mathrm{sm}}, \theta_{\mathrm{sm}}\}$. Here, $k = k_1 = k_2$; w_i and w_k are the plate displacements at points where the 2-dof spring–mass system is mounted; and w_{sm} and θ_{sm} are the displacement and rotation of the attached mass.

4. Numerical results

Numerical results are presented for the cases where the plate is simply supported on all its sides in Table 1 and for a cantilever plate, clamped at $x = 0$ and free on the remainder, in Table 2. The

Table 1
First eight natural frequencies of simply supported square plate carrying a 2-dof spring–mass system located at $x_1 = 0.2,\ y_v = 0.5$ and $x_2 = 0.4,\ y_v = 0.5$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
Present	56.7879	556.9301	1542.1984	3779.8722	3790.8440	6047.7956	7565.1100	7565.1610
FEM	56.7315	556.4749	1542.1403	3779.8807	3790.8311	6047.7665	7565.2602	7565.3048
Bare plate	—	—	1511.9489	3779.8722	3779.8722	6047.7956	7559.7445	7559.7445
2-dof	56.8885	569.5240	—	—	—	—	—	—

Note: 2-dof: natural frequencies of the two degree of freedom system attached to the plate.

Table 2
First eight natural frequencies of a cantilever square plate carrying a 2-dof spring–mass system located at $x_1 = 0.2,\ y_v = 0.5$ and $x_2 = 0.4,\ y_v = 0.5$

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
Present	56.4650	270.0592	567.9746	661.4260	1666.9924	2116.6842	2407.1138	4220.9719
FEM	56.3894	270.0388	566.7005	661.3694	1661.8238	2121.9127	2406.7592	4219.6910
Bare plate	—	269.8571	—	661.4260	1654.9301	2115.2492	2407.1138	4214.5193
2-dof	56.8885	569.5240	—	—	—	—	—	—

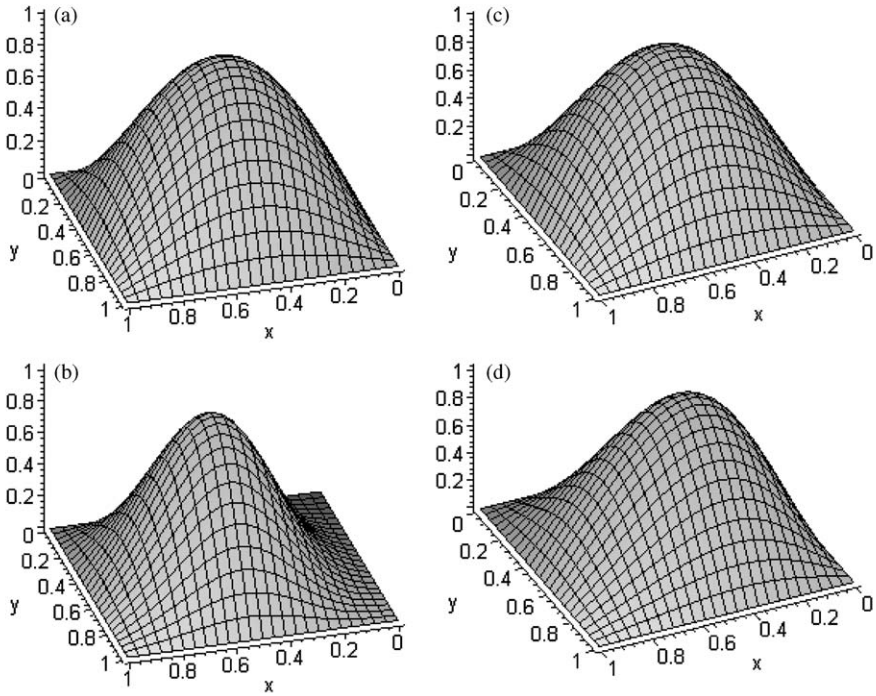

Fig. 2. Modes of the simply supported plate: (a) first mode of the bare plate, (b) first mode of the modified plate (plate + 2-dof system attached), (c) first mode of the bare plate for comparison with the second mode of the modified system, (d) second mode of the modified system.

mode shapes of the plate are represented in Fig. 2 for the simply supported plate and in Fig. 3 for the cantilever plate.

In Fig. 2(a) the first mode for the bare plate is illustrated, with its corresponding first mode of the modified system, in Fig. 2(b). In Fig. 2(c) we present, for comparison, the first mode of the bare plate because it is very similar to the second mode of the modified system (plate with the attached discrete system) shown in Fig. 2(d). The other modes will not be presented here due to the similarity between the bare and modified plate modes. The same occurs for the cantilever plate. In Fig. 3(a) the first mode of the bare structure is presented with its corresponding first mode of the modified system in Fig. 3(b). We realize that the second frequency of the cantilever plate (Table 2) possesses practically the same values for the modified and unmodified system.

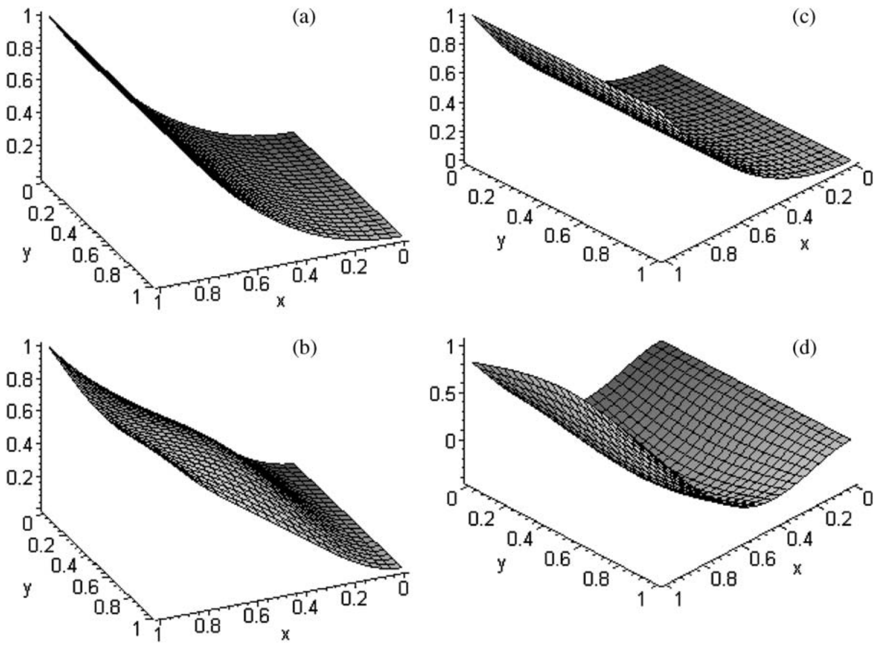

Fig. 3. Modes of the cantilever plate: (a) first mode of the bare plate, (b) first mode of the modified plate, (c) first mode of the bare plate for comparison with the third mode of the modified system, (d) third mode of the modified system (mode two is not shown here because it remains the same as for the unmodified system).

Accordingly their modes are very similar. On the other hand we present in Fig. 3(d) the third mode of the modified system and the first mode of the bare plate in Fig. 3(c) for comparison.

The dimensions and material constants for the plate are: length $a = 1\,\mathrm{m}$, width $b = 1\,\mathrm{m}$, thickness $h = 0.05\,\mathrm{m}$, mass density $\rho = 7.8367 \times 10^3\,\mathrm{kg/m^3}$, mass plate $m_p = \rho hab = 391.835\,\mathrm{kg}$, Poisson coefficient $v = 0.3$, Young modulus $E = 2.069 \times 10^{11}\,\mathrm{N/m^2}$.

The 2-dof spring–mass system is located at $x_1 = 0.2\,\mathrm{m}$ and $x_2 = 0.4\,\mathrm{m}$ with $a_1 = 0.06667\,\mathrm{m}$ and $a_2 = 0.13333\,\mathrm{m}$ as it is presented in Fig. 1. For the mass added at x_1, y_v and x_2, y_v we have $m_e = 0.1 \times m_p\,\mathrm{kg}$, $I_e = 0.1 \times m_p \times ab\,\mathrm{kg\,m^2}$, $k_1 = 6.34761 \times 10^6\,\mathrm{N/m}$, $k_2 = k_1$. These dimensions were chosen to make a correlation between present results and those previously published [6,7].

The normal mode shape functions, used for the calculations leading to the results given in Figs. 2 and 3, will be detailed in Appendix A. The modes chosen for the simply supported and the cantilever plate were the first six modes.

5. Conclusions

In this paper the natural frequencies and mode shapes of a plate carrying a 2-dof system were calculated. The technique presented here is simpler than those previously presented by other methods, for example in Refs. [6,7], because our calculations lead to a determinant of order 2 despite the number of modes that have been used to create the transverse displacement function in Eq. (4).

Additionally, an alternative way to determine the results was presented employing the FEM method (Section 3). By using this scheme the reliability of the analytical results were satisfactorily confirmed.

The plate modes were drawn for the simply supported (Fig. 2) and for the cantilever plate (Fig. 3). It is clearly noticed that, for that mode where the mounted system is just in the position of a nodal line of the 'bare' structure, the plate seems to be undisturbed and the mode remains the same. This fact can also be seen from Tables 1 and 2 where, for the modes 4 and 6 in the case of the simply supported plate, and 4 and 7 in the case of the cantilever plate, the frequency also remains invariant.

An interesting conclusion could be deduced from Tables 1 and 2. It can be observed that all the frequencies of the 2-dof-plate system, higher than the second are increased because of the effect that the addition of the 2-dof system exerts on the plate. This fact could be understand from the well known results due to Rayleigh [8], synthesized by Dowell [2] in the following statement: "If a spring–mass combination (which by itself has a rigid body as well as an elastic degree of freedom) is added to another system, the frequencies originally higher than the basic spring–mass frequency are increased, those originally lower are decreased, and a new frequency appears between the originally pair of frequencies nearest the spring–spring frequency". This result could be applied for this case since the 2-dof may be viewed as two systems of 1-dof (obviously coupled) of frequencies lower than the original system (bare plate). So the net effect is to raise the resultant frequencies of the combined system.

With regards to the selection of the number of modes assumed for the displacement amplitude, Eq. (4), we must clarify that in the case of the simply supported plate, the number of modes were six. This fact is properly justified in the sense that, if one considers a major number of modes, the changes in all the frequency values are always within the 0.05% of the presented values (Table 1). So adding more modes in the calculations produces no significative modification in the frequencies values but increases the computational time and effort instead. The same is observed in the case of the cantilever plate case where also six modes of the bare plate have been used.

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Appendix A

The normal mode shape functions used for the calculations are, for simply supported plate,

$$\phi_{ij}(x,y) = \sin\left(\frac{i\pi x}{a}\right)\sin\left(\frac{j\pi y}{b}\right), \tag{A.1}$$

and for the cantilever plate,

$$\phi_{ij}(x,y) = \Phi_i(x)\Psi_j(y), \tag{A.2}$$

where $\Phi_i(x)$ and $\Psi_j(y)$ are beam functions which satisfy clamped–free boundary conditions and free–free conditions, respectively, and are defined as

$$\Phi_i(\zeta) = \mu_i(\cosh \alpha_i\zeta - \cos \alpha_i\zeta) - \nu_i(\sinh \alpha_i\zeta - \sin \alpha_i\zeta) \quad (i = 1, 2, 3, \ldots), \tag{A.3}$$

where $\zeta = x/a$,

$$\mu_i = \frac{\cosh \alpha_i + \cos \alpha_i}{\sinh \alpha_i \sin \alpha_i}, \quad \nu_i = \frac{\sinh \alpha_i - \sin \alpha_i}{\sinh \alpha_i \sin \alpha_i}, \tag{A.4}$$

and α_i are the roots of

$$\cosh \alpha_i \cos \alpha_i = -1. \tag{A.5}$$

The first three roots of Eq. (A.5) are $\alpha_1 = 1.875$, $\alpha_2 = 4.694$ and $\alpha_3 = 7.854$.

For the free–free function $\Psi_j(\eta)$ $(\eta = y/b)$, we have $(j = 3, 4, 5 \ldots)$

$$\Psi_1(\eta) = 1, \quad \Psi_2(\eta) = \sqrt{3}(2\eta - 1), \tag{A.6}$$

$$\Psi_j(\eta) = \xi_j(\cosh \beta_j\eta + \cos \beta_j\eta) - \chi_j(\sinh \beta_i\eta + \sin \beta_j\eta), \tag{A.7}$$

$$\xi_j = \frac{\cosh \beta_j - \cos \beta_j}{\sinh \beta_j \sin \beta_j}, \; \chi_j = \frac{\sinh \beta_j + \sin \beta_j}{\sinh \beta_j \sin \beta_j}, \tag{A.8}$$

and β_j $(j \geqslant 3)$ are the roots of

$$\cosh \beta_j \cos \beta_j = 1 \tag{A.9}$$

where $\beta_3 = 4.730$, $\beta_4 = 7.853$ and $\beta_5 = 10.995$.

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